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Pseudo-invariant Eigen-Operator for Deriving Energy-Level Gap for Quantum Bit in Quantum Circuit

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Abstract Based on the method of pseudo invariant eigenoperator (PIEO), a fully quantum mechanical scheme is investigated for the coupling between a rf SQUID qubit and an off-resonance quantized single-mode electromagnetic field in the strong coupling regime. In order to derive the systematic energy-level gap obtained by the pseudo-invariant operator of the quantum system, we give operation props for corresponding quantum manipulation. In comparison with the solution of stationary Schrödinger equation, the PIEO method could be quite concise and effective to obtain energy-level gap for the given system.

Keywords Heisenberg equation · Pseudo invariant eigen-operator method · Superconducting qubit · Energy gap

1 Introduction

Significant progress has been made on physical implementation of quantum computation based on superconducting qubits [1–4]. Quantum coherence has been successfully demonstrated in a variety of superconducting single-qubit systems and coupled two-qubit systems, which indicate a potential application of superconducting qubits in quantum computing. Superconductor qubit is macroscopical quantum qubit with a large size and many merits. Especially, it can be easily controlled, coupled and measured.

During the quantum information processing (including quantum computation and quantum manipulation), one need to know the exact level or energy-level gap for the quantum system so as consequently to choose the magnitude and frequency of corresponding external control signal. By solving various stationary Schrödinger equations, generally, one

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may get the eigenvalues and eigenvectors of dynamic Hamiltonians. Moreover, though the Heisenberg equation stands on the same footing as the Schrödinger equation, it is seldom employed for the purpose of directly deriving the energy eigenvalues. In very recent papers [5–7] authors has introduced a new method, i.e. the "invariant eigen-operator" (IEO) method to explore energy-level gap of dynamic Hamiltonians, which is based on the concept of both Schrödinger operator and the Heisenberg equation of motion. In [8, 9], the authors further extend this method to pseudo-invariant eigen-operator.

In this manuscript, we apply the method of pseudo invariant eigen-operator to investigate the superconducting single qubit and double qubits system. We obtain not only the pseudo invariant eigen-operator but also the energy-level gap by a different method.

2 The Hamiltonian of a rf SQUID Qubit

Josephson junction is the core component for superconducting qubit. If the number of the Cooper pair in tunnel junction is used as variable, the superconducting charge qubit is obtained; if the phase difference Φ of Josephson junction is used as displacement variable, then the superconducting phase qubit is formed; If we connect both ends of a Josephson junction with a superconducting line, we obtain a superconducting loop with a Josephson junction, which is known as RF SQUID forming superconducting flux qubit. In charge qubit, the Josephson energy E_J is less than the charge energy in Josephson junction $E_C = e^2/2C_J$. The condition is opposite for the two subsequent cases. Under whatever case, all of them can be seen as a spin system in two-level approximation. In order to implement information processing, the qubit system should interact with the external quantum radiation field [10–12].

Let us consider a fully quantum mechanical scheme for the coupling between a rf SQUID qubit and an off-resonance quantized single-mode electromagnetic field in the strong coupling regime. For the sake of generality, we model this monochromatic field as a $L_F C_F$ resonator. The effective Hamiltonian of the system under the rotating approximation is [13–15]

$$\hat{H}_S = \frac{1}{2}\omega_0\hat{\sigma}_z + \omega_F\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) + g(\hat{a}^+\hat{\sigma}_- + \hat{a}\hat{\sigma}_+), \tag{1}$$

where $\hat{a}^+$ and $\hat{a}$ denote the photon creation and annihilation operators with the commutation relation $[\hat{a}, \hat{a}^+] = 1$. ω_0 is an inverse of resonance frequency of qubit, ω_F is a radioactive field frequency, and g is a coupling strength between the radioactive field and single-qubit system. $\hat{\sigma}_+(\hat{\sigma}_-)$ are the pseudospin operators for qubit defined as $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$ with $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are the Pauli matrices, i.e.,

$$\hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

obeying the following relations,

$$[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \qquad [\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z. \tag{2}$$

By defining the tensor operators

$$\hat{V} = \begin{pmatrix} \hat{a}^+\hat{a} + 1 & 0 \\ 0 & \hat{a}^+\hat{a} \end{pmatrix}, \tag{3}$$

$$\hat{M} = \begin{pmatrix} -\hat{a}^+\hat{a} & 0 \\ 0 & -\hat{a}^+\hat{a} - 1 \end{pmatrix}, \tag{4}$$

$$\hat{Q}_+ = \begin{pmatrix} 0 & 0 \\ \hat{a}^+ & 0 \end{pmatrix}, \qquad \hat{Q}_- = \begin{pmatrix} 0 & \hat{a} \\ 0 & 0 \end{pmatrix}. \tag{5}$$

The Hamiltonian is rewritten as

$$\hat{H}_S = \frac{1}{2}(\omega_0 + \omega_F)\hat{V} + \frac{1}{2}(\omega_0 + \omega_F)\hat{M} + g(\hat{Q}_+ + \hat{Q}_-), \tag{6}$$

where $\hat{V}$ is a Casimir operator. The set of operators $\{\hat{V}, \hat{M}, \hat{Q}_+, \hat{Q}_-\}$ generates a dynamically closed superalgebra and satisfy the following commutation and anticommutation relations

$$\{\hat{Q}_\varepsilon, \hat{Q}_\eta\} = \frac{1}{2}\hat{V}\delta_{\varepsilon,-\eta}, \tag{7}$$

$$[\hat{V}, \hat{M}] = [\hat{V}, \hat{Q}_\varepsilon] = 0, \tag{8}$$

$$[\hat{M}, \hat{Q}_\varepsilon] = -2\varepsilon\hat{Q}_\varepsilon, \tag{9}$$

$$\{\hat{Q}_\varepsilon, \hat{\sigma}_z\} = 0 \quad (\varepsilon, \eta = \pm 1) \tag{10}$$

According to the invariant eigen-operator method, we should firstly consider the basic commutative relations using the Heisenberg equation

$$[\hat{V}, \hat{H}_S] = 0, \tag{11}$$

$$[\hat{M}, \hat{H}_S] = -2\sqrt{2}g(\hat{Q}_+ - \hat{Q}_-), \tag{12}$$

$$[\hat{Q}_+, \hat{H}_S] = (\omega_0 - \omega_F)\hat{Q}_+ + \sqrt{2}g\hat{F}, \tag{13}$$

$$[\hat{Q}_-, \hat{H}_S] = -(\omega_0 - \omega_F)\hat{Q}_- - \sqrt{2}g\hat{F}, \tag{14}$$

where

$$\hat{F} = [\hat{Q}_+, \hat{Q}_-] = \hat{Q}_+\hat{Q}_- - \hat{Q}_-\hat{Q}_+.$$

3 Pseudo Invariant Eigenoperator for the rf SQUID Qubit

In this section we shall apply the pseudo invariant eigen-operator method to derive the energy level of the system. Based on Hamiltonian $\hat{H}_S$ in (6), we suppose that the invariant eigen-operator is (see Appendices in detail)

$$\hat{Q}_{se} = \alpha + \beta(\hat{Q}_+ + \hat{Q}_-). \tag{15}$$

The parameters α and β will be determined in the following.

With the help of (11)–(14), one gets

$$i\hbar\frac{d}{dt}\hat{Q}_{se} = \alpha\Delta(\hat{Q}_+ - \hat{Q}_-),$$

where

$$\Delta = \omega_0 - \omega_F$$

Furthermore, we have

$$\left(i\hbar\frac{d}{dt}\right)^2\hat{Q}_{se}=\alpha\Delta^2(\hat{Q}_++\hat{Q}_-)+2\sqrt{2}g\Delta\hat{F} \tag{16}$$

Comparing the right-hand side of (16) with one of (15), we see that $\alpha\Delta^2(\hat{Q}_++\hat{Q}_-)+2\sqrt{2}g\Delta\hat{F}$ is impossibly proportional to $\hat{O}_{se}$. However, we can easily verify $[\hat{F},\hat{H}_S]=0$. Namely, $\hat{F}$ and $\hat{H}_S$ have the same eigenstates. The eigenstates of $\hat{F}$ and $\hat{H}_S$ are

$$\begin{pmatrix}|n\rangle\\0\end{pmatrix}=|n\rangle\otimes|1\rangle,\qquad \begin{pmatrix}0\\|n+1\rangle\end{pmatrix}=|n+1\rangle\otimes|0\rangle,$$

which are a direct product of the Fock state $|n\rangle$ and the pseudospin states. It is easily to obtain the following relations,

$$\hat{F}\begin{pmatrix}|n\rangle\\0\end{pmatrix}=(n+1)\begin{pmatrix}|n\rangle\\0\end{pmatrix}$$
$$\hat{F}\begin{pmatrix}0\\|n+1\rangle\end{pmatrix}=(n+1)\begin{pmatrix}0\\|n+1\rangle\end{pmatrix}$$

Substitute (16) to the common eigenstate of operators $\hat{F}$ and $\hat{H}_S$ respectively, we find

$$\left(i\frac{d}{dt}\right)^2\hat{O}_{se}\begin{pmatrix}|n\rangle\\0\end{pmatrix}=\alpha[\Delta^2(\hat{Q}_++\hat{Q}_-)-2\sqrt{2}g(n+1)]\begin{pmatrix}|n\rangle\\0\end{pmatrix}, \tag{17}$$

$$\left(i\frac{d}{dt}\right)^2\hat{O}_{se}\begin{pmatrix}0\\|n+1\rangle\end{pmatrix}=\alpha[\Delta^2(\hat{Q}_++\hat{Q}_-)-2\sqrt{2}g(n+1)]\begin{pmatrix}0\\|n+1\rangle\end{pmatrix}. \tag{18}$$

From above equations, we know, in the eigenvector space of$\hat{F}$, that (17) and (18) can be comparable with $\hat{O}_{se}$ in (15).

If the operator $\hat{Q}_{se}$ satisfies (A.2), by using (3)–(5), we are bound to have

$$\frac{\alpha}{\beta}=\frac{\Delta}{2\sqrt{2}g(n+1)}. \tag{19}$$

As a consequence of (15), the invariant eigen-operator of the system is expressed as

$$\hat{Q}_{se}=\alpha\left[1+\frac{2\sqrt{2}g(n+1)}{\omega_0-\omega_F}(\hat{Q}_++\hat{Q}_-)\right]. \tag{20}$$

Expression (20) indicates that, for the case of an uncertain constant α, there are multi-pseudo invariant eigen-operators in this quantum system. Similar situation also exists in other systems, which offer us more convenience.

From (16) and (18), we can express the level gap as

$$G=\sqrt{(\omega_0-\omega_F)^2+4g^2(n+1)}. \tag{21}$$

From (21), we are readily to see that, for single qubit in quantum radiation field, the system energy-level gap not only relies on the frequencies of the qubit and radiation field, but also related closely on the coupling strength between qubit and radiation field. Furthermore, we may know as well that the energy-level gap decreases continuously with the increasing of the energy-level, which practically reflects in profile that it is easier to carry out exact quantum manipulation by means of the lower energy-levels of the system.

In quantum logic operations and computations, it is necessary to discuss coupled two-qubit dynamic evolution. In order to simplify our computation but without loss of generality, we only consider the Casimir interaction between the control and target qubits. The total Hamiltonian is [16]

$$\hat{H}_C = \frac{1}{2}\omega_0\sigma_{1z} + \omega_F\left(a^+a + \frac{1}{2}\right) + g(a^+\sigma_{1-} + a\sigma_{1+}) + \lambda V_1 V_2. \tag{22}$$

Based on the Hamiltonian $\hat{H}_C$, we suppose that the invariant eigen-operator in the case is

$$\hat{Q}_{ce} = \alpha + \beta(\hat{Q}_{1+} + \hat{Q}_{1-}) + \gamma V_1 V_2. \tag{23}$$

By using the following relations,

$$[A, BC] = [A, B]C + B[A, C],$$
$$[AB, C] = [A, C]B + A[B, C]$$

after substituting (23) into (A.2) and combing (11)–(14), we have

$$\left(i\hbar\frac{d}{dt}\right)^2 \hat{Q}_{ce} = \alpha\Delta^2(\hat{Q}_{1+} + \hat{Q}_{1-}) + 2\sqrt{2}g\Delta\hat{F}_1, \tag{24}$$

where

$$\hat{F}_1 = \hat{Q}_{1+}\hat{Q}_{1-} - \hat{Q}_{1-}\hat{Q}_{1+}.$$

Comparing the right-hand side of (24) with one of (23), we find that $\alpha\Delta^2(\hat{Q}_{1+} + \hat{Q}_{1-}) + 2\sqrt{2}g\Delta\hat{F}_1$ is not proportional to $\hat{O}_{ce}$. However, we surprise to find the relation $[\hat{F}_1, \hat{H}_C] = 0$, i.e., $\hat{F}_1$ and $\hat{H}_C$ have the same eigenstates given by

$$|00\rangle = \begin{pmatrix} |n,n\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |01\rangle = \begin{pmatrix} 0 \\ |n,n+1\rangle \\ 0 \\ 0 \end{pmatrix},$$

$$|10\rangle = \begin{pmatrix} 0 \\ 0 \\ |n+1,n\rangle \\ 0 \end{pmatrix}, \qquad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ |n+1,n+1\rangle \end{pmatrix},$$

which are the direct product of the Fock state $|n\rangle$ and the pseudospin states. It is easy to deduce the following equations,

$$\left(i\frac{d}{dt}\right)^2 \hat{O}_{ce}\begin{pmatrix} |n,n\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix} = \alpha[\Delta^2(\hat{Q}_{1+} + \hat{Q}_{1-}) - 2\sqrt{2}g(n+1)]\begin{pmatrix} |n,n\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{25}$$

$$\left(i\frac{d}{dt}\right)^2 \hat{O}_{ce}\begin{pmatrix}0\\|n+1,n\rangle\\0\\0\end{pmatrix} = \alpha[\Delta^2(\hat{Q}_{1+}+\hat{Q}_{1-})-2\sqrt{2}g(n+1)]\begin{pmatrix}0\\|n+1,n\rangle\\0\\0\end{pmatrix}. \quad (26)$$

Utilizing the two upper expressions and combined with (23) and (A.2), we may conclude that the coupling qubit system has the completely same pseudo invariant eigen-operator and energy gap with the single qubit system.

In summary, we have adopted the invariant eigen-operator method to tackle the superconducting single qubit and double qubits system. This approach seems to be concise and direct so as to be extended to the other Hamiltonian models.

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Appendix A: Appendix The Invariant Eigen-Operator Method

In this appendix, we briefly review the invariant eigen-operator operator method. In quantum mechanics, for a system with the Hamiltonian $\hat{H}$ and $\frac{\partial \hat{H}}{\partial t}=0$, which does not contain the time, its stationary Schrödinger equation is $i\dot{\psi}=\hat{H}\psi=E\psi$, $(h=1)$. While in the Heisenberg picture, the dynamic evolution of time of the operator $\hat{O}$ is dominated by the Heisenberg equation

$$i\frac{d\hat{O}}{dt}=[\hat{O},\hat{H}]. \quad (A.1)$$

If the operator satisfies

$$\left(i\frac{d}{dt}\right)^n \hat{O}_e=[\ldots[[\hat{O}_e,\hat{H}],\hat{H}],\ldots]=\lambda\hat{O}_e, \quad n\geq 2 \quad (A.2)$$

similarly to Schrödinger equation $\hat{H}\psi=E\psi$, (A.1) and (A.2) are operator equations with the eigenvector $\hat{O}_e$ and the stationary eigenvalue is λ. Therefore, we call (A.2) as the operator equation of invariant eigen-operator operator method. Furthermore, one may find that the eigenvalue $\sqrt[n]{\lambda}$ is the energy gap of the quantum system.

Appendix B: The Pseudo Invariant Eigen-Operator Method

For some quantum systems, the operator $\hat{O}_e$ that is composed of several suboperators satisfied $[[\ldots,[[\hat{O}_e,\hat{H}]],\ldots],\hat{H}]=\lambda\hat{O}'_e$. But $\hat{O}_e$ and $\hat{O}'_e$ are disproportional.

At the same time, if one of the operators $\hat{F}$ of $\hat{O}'_e$ has the same eigenstates with the Hamiltonian $\hat{H}$, i.e.,

$$[\hat{F},\hat{H}]=0 \quad (B.1)$$

in this common eigenstate space $|\psi\rangle$, the operator $\hat{F}$ has the degenerated eigenvalue so that $\hat{O}_e$ and $\hat{O}'_e$ are to proportional. The operator $\hat{O}_e$ is called as the pseudo invariant eigen-operator (PIEO) of system. The operator $\hat{O}_e$ satisfies the following equation

$$\left(i\frac{d}{dt}\right)^n \hat{O}_e|\psi\rangle=\lambda\hat{O}_e|\psi\rangle, \quad (B.2)$$

while $\sqrt[n]{\lambda}$ describes the energy gap of system.

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